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AN AGGREGATED MODEL
OF A TWO - SIDED NUCLEAR WAR

John Donelson, III

September 1975



INSTITUTE FOR DEFENSE ANALYSES
PROGRAM ANALYSIS DIVISION

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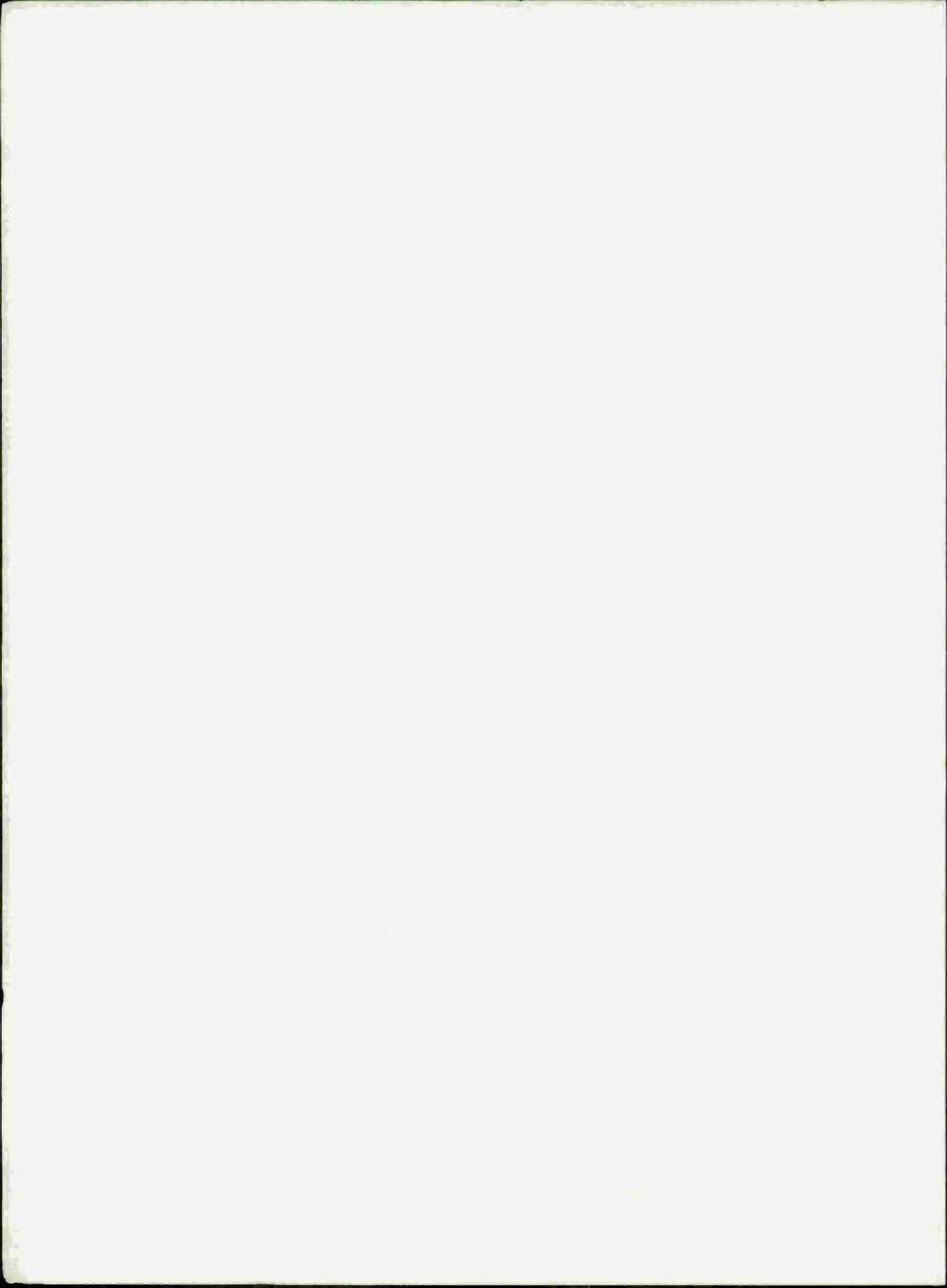
John Donelson, III

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in this paper is highly aggregated. Value targets on each side are treated as a single target for the other side. The strategic weapons of the second side are also treated as a single weapon target for the first striker's counterforce attack. Thus, the weapon allocations determined by the model are also aggregated. The two-sided nuclear war is then posed as a max-min problem in which the first striker seeks to maximize the chosen measure of effectiveness, while the second striker tries to minimize the measure with his surviving resources. Various random effects are also included in the model.

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PREFACE

AN AGGREGATED MODEL OF A TWO-SIDED NUCLEAR WAR

John Donelson III

In this paper we consider the problem of modeling a two-sided nuclear war. In such a war, the first striker (denoted by A) is faced with the problem of allocating his weapons against the value targets and weapon targets of the second striker (denoted by B). There are a number of important variables that would in reality affect A's weapon allocation against B (e.g., the yield, reliability, and accuracy of A's warheads; the hardness of B's targets; and the total number of weapons on both sides); but one of the most important--which affects A's first-strike allocation--is the capability of B to retaliate after absorbing a first strike by A. Thus, A must somehow take account of B's retaliatory capability, and the manner in which A does so in the decision-making process is very important. If B's strategic weapons are somewhat vulnerable to a first strike by A, then, by attacking B's weapons in a massive way, A may be able to minimize the retaliatory damage that B can inflict. On the other hand, either if A's warheads are of low accuracy or if B's strategic weapon targets are relatively invulnerable, then A may be unable to destroy very many of B's strategic weapons. In this case, A would almost certainly be faced with a massive retaliatory strike by B.

The model that we shall develop in this paper is quite simple. All value targets are aggregated into a single value target. B's weapon targets are treated as a single class of weapon targets for A to strike against. The weapon allocations determined by the model are also aggregated. The model

allocates a block of A's warheads against B's value targets and another block of weapons against B's weapon-launchers. However, we are able to consider a sufficient number of real-world effects to make the model interesting and important.

There are several advantages of a two-sided model of nuclear war, as opposed to those models that treat one-sided strikes. In reality, A would have to consider the variables that affect B's ability to retaliate successfully. One-sided models of a nuclear strike do not correctly take account of this interaction. In a two-sided model, however, the interactions between the variables affecting the strategic capabilities of both sides are built into the model. Another advantage of a two-sided model is that it does not require any assignment of intrinsic value to strategic weapons. A's allocation of warheads to destroy B's strategic weapons is determined by the ability of B's weapons to inflict retaliatory damage--not by their intrinsic value relative to other value targets.

In Chapter I (below), we derive the basic model and develop the appropriate measures of effectiveness. These derivations are done under the assumption that the variables interact in a deterministic manner.

In reality, there is randomness in the number of B weapons that survive A's first strike. There is also additional randomness introduced by the reliability and accuracy of the weapons on both sides. In Chapter II, we consider some of these effects and perform the calculations for some special cases. We also consider other technical issues in Chapter II.

In Chapter III, we derive a damage law for the economy of the United States using data derived from a real-world data base. The procedure used in Chapter III is reasonable and simple, and it should be applicable to many data bases.

Finally, in Chapter IV, we consider several numerical examples--including a hypothetical nuclear war between the United States and the Soviet Union. These examples illustrate the use of the model and also show the sensitivity of the allocations and damage levels to various parameters that occur in the model.

Chapter I

DERIVATION OF THE DETERMINISTIC MODEL

We begin with the assumption that nation A has n nuclear warheads and decides to launch a first strike against nation B. B has m weapon-launchers, each of which is capable of delivering μ warheads. B's weapon-launchers constitute a possible class of weapon targets for A. Let v_a denote the value of A's economy and let v_b denote the value of B's economy. (In this paper, v_a and v_b are value added.) We assume that B's economy constitutes a single value target for A having value v_b . Similarly, A's economy constitutes a single aggregated target having value v_a , against which B retaliates.

In this chapter, we assume that all of A's warheads have the same yield, which we shall not specify at this time. We also assume that A's warheads are perfectly reliable, so that each warhead allocated by A functions correctly and detonates on its target. (We make similar assumptions for B's strategic weapons.)

A. CONDITIONAL DAMAGE RATE AND DERIVATION OF THE DAMAGE LAW

Let $B(x)$ denote the fraction of B's economy surviving after a first strike by A with x ($\leq n$) warheads. The assumption that A's warheads have reliability 1.0 implies that each warhead allocated by A detonates on or near its target. Since $B(x)$ is decreasing for $x \geq 0$, its first derivative--denoted by $B'(x)$ --is negative. We define the *conditional damage rate* to B's economy--denoted by $b(x)$ --to be the negative logarithmic derivative of $B(x)$. That is,

$$b(x) = -\frac{d}{dx} \ln B(x) = -\frac{B'(x)}{B(x)}. \quad (1)$$

Thus, $b(x)dx$ represents the fraction of B's remaining economy that would be destroyed by dx additional warheads, given that x warheads have already detonated against B's economy. Since $B(x)$ is a fraction, we have $0 \leq B(x) \leq 1$ for $x \geq 0$. Also, we have the initial condition $B(0) = 1$, since there is no damage to B's economy if $x = 0$. Integrating both sides of Equation (1) over the interval $[0, x]$ and using the initial condition, we deduce that

$$B(x) = \exp\left(-\int_0^x b(u)du\right) \quad (2)$$

for $x \geq 0$.

Let $D_B(x)$ denote the damage to B's economy caused by detonating x warheads. Then, for $x \geq 0$, we have

$$D_B(x) = v_b[1-B(x)] = v_b\left[1 - \exp\left(-\int_0^x b(u)du\right)\right]. \quad (3)$$

For example, when $b(u) \equiv b$ ($b > 0$) for $u \geq 0$, we see that $B(x) = e^{-bx}$ and that $D_B(x) = v_b(1-e^{-bx})$.

Let $A(y)$ denote the fraction of A's economy surviving after a retaliatory attack by B with y warheads. If we let $a(y)$ denote the conditional damage rate for A's economy, then (as before) we have

$$A(y) = \exp\left(-\int_0^y a(u)du\right) \quad (4)$$

for $y \geq 0$. Let $D_A(y)$ denote the damage inflicted on A's economy in a nuclear attack using y warheads. Then

$$D_A(y) = v_a[1-A(y)] = v_a\left[1 - \exp\left(-\int_0^y a(u)du\right)\right] \quad (5)$$

for $y \geq 0$.

B. RETALIATORY DAMAGE TO A

In a two-sided exchange, if A is able to destroy a sizable portion of B's strategic arsenal, the retaliatory damage to A may be limited to some extent. The most important variable affecting B's ability to retaliate against A is the size of B's surviving arsenal. In reality, there are a number of physical variables that affect A's ability to damage or destroy B's strategic arsenal. Though we consider some of these effects in Section 3, we limit our consideration in this section to finding an expression that determines the number of B-weapon-launchers surviving as a function of the size of A's attack on B's strategic arsenal.

Let $C(w)$ denote the fraction of B's weapon-launchers surviving after an attack by w ($\leq n$) warheads. If $c(w)$ is the conditional damage rate for A's warheads attacking B's weapon-launchers, then we have

$$C(w) = \exp\left(-\int_0^w c(u)du\right)$$

for $w \geq 0$.

If B has m weapon-launchers initially, then the number of B-weapon-launchers surviving an attack by w warheads--denoted by $N(w)$ --is given by

$$N(w) = mC(w) = m \exp\left(-\int_0^w c(u)du\right). \quad (6)$$

If each B-weapon-launcher is capable of delivering μ warheads, then B is able to retaliate with a total of

$$\mu N(w) = \mu m \exp\left(-\int_0^w c(u)du\right) \quad (7)$$

warheads.

In order to compute the maximum retaliatory damage that B can inflict on A, we note that $D_A(y)$ is increasing in y . Hence, B can maximize retaliatory damage by allocating all of its surviving weapons against A's economy. Thus, when A allocates w weapons to counterforce operations against B's strategic arsenal, the maximum retaliatory damage that B can inflict on A is given by the expression

$$D_A(\mu N(w)) = v_a \left[1 - \exp \left(- \int_0^{\mu N(w)} a(u) du \right) \right], \quad (8)$$

where $N(w)$ is given by equation (6).

C. MEASURES OF EFFECTIVENESS

Since A plans to strike first, its leaders must decide how to allocate their n warheads against B's value targets and weapon targets. Let x be the number of warheads allocated by A against B's value targets; w , the number of warheads allocated by A against B's weapon-launchers. Then $x + w \leq n$, since A has only n warheads. By striking at B's strategic weapons, A hopes to reduce B's capacity to retaliate.

The choice of A's allocation (x, w) will depend upon the measure of effectiveness (MOE) that is used. We consider two MOEs for evaluating A's allocation. Since the exchange is two-sided (in the sense that A attacks first and B then retaliates), the problem may be stated as a max-min problem in which the first striker (A) uses his resources to maximize the MOE and the second striker (B) uses his surviving weapons to minimize the MOE.

The first MOE that we examine is the difference between the damage to B and the retaliatory damage to A. For this MOE, the two-sided exchange can be stated as a max-min problem of the form

$$\max_{\substack{x, w \geq 0 \\ x+w \leq n}} \min_{y \leq \mu N(w)} [D_B(x) - D_A(y)] .$$

Since $D_A(y)$ is increasing in y , the inner problem is solved by setting $y = \mu N(w)$. Hence, the max-min problem reduces to the following one-sided optimization problem:

$$\max_{\substack{x, w \geq 0 \\ x+w \leq n}} \left[D_B(x) - D_A(\mu N(w)) \right]. \quad (9)$$

Let $D(x, w)$ denote the function to be maximized in (9). Then, by substituting Equations (3) and (5) into (9), we get

$$D(x, w) = D_B(x) - D_A(\mu N(w)) = v_b \left[1 - \exp\left(-\int_0^x b(u) du\right) \right] - v_a \left[1 - \exp\left(-\int_0^{\mu N(w)} a(u) du\right) \right], \quad (10)$$

where $N(w)$ is given by Equation (6). It is easy to show that $D(x, w)$ is increasing in both x and w , since $N(w)$ is decreasing in w . This fact implies that the maximum in (9) occurs at a point (x, w) that satisfies the constraint $x + w = n$. Therefore, in order to solve (9), it suffices to find x in the interval $[0, n]$, which maximizes $D(x, n-x)$. Making use of Equation (10), the optimization problem in (9) reduces to the following one-dimensional optimization problem:

$$\text{maximize: } v_b \left[1 - \exp\left(-\int_0^x b(u) du\right) \right] - v_a \left[1 - \exp\left(-\int_0^{\mu N(n-x)} a(u) du\right) \right] \quad x \in [0, n]. \quad (11)$$

Thus, the two-sided max-min problem is reduced to a one-dimensional optimization problem in which A seeks to maximize the damage to B minus the *maximum* retaliatory damage to A.

The second MOE that we consider is the ratio of the damage to B to the retaliatory damage to A. In this case, the attack and retaliation model can be stated as the following max-min problem:

$$\max_{\substack{x, w \geq 0 \\ x+w \leq n}} \min_{y \leq \mu N(w)} \frac{D_B(x)}{D_A(y)} .$$

Again, since $D_A(y)$ is increasing in y , the inner problem is solved by setting $y = \mu N(w)$; and the max-min problem reduces to the following optimization problem:

$$\max_{\substack{x, w \geq 0 \\ x+w \leq n}} \frac{D_B(x)}{D_A(\mu N(w))} . \quad (12)$$

Thus, the MOE--denoted in this case by $R(x, w)$ --is given by

$$R(x, w) = \frac{D_B(x)}{D_A(\mu N(w))} = v_b v_a^{-1} \left[1 - \exp\left(-\int_0^x b(u) du\right) \right] \left[1 - \exp\left(-\int_0^{\mu N(w)} a(u) du\right) \right]^{-1} . \quad (13)$$

Since $R(x, w)$ is increasing in both x and w , the maximum in (12) occurs at a point (x, w) that satisfies $x + w = n$. Hence, in order to solve (12), it suffices to find x in the interval $[0, n]$, which maximizes

$$\begin{aligned} R(x) = R(x, n-x) &= \frac{D_B(x)}{D_A(\mu N(n-x))} \\ &= v_b v_a^{-1} \left[1 - \exp\left(-\int_0^x b(u) du\right) \right] \left[1 - \exp\left(-\int_0^{\mu N(n-x)} a(u) du\right) \right]^{-1} . \end{aligned} \quad (14)$$

If x^* maximizes $R(x)$ in the interval $[0, n]$, then x^* and $w^* = n - x^*$ solves the problem stated in (12).

In finding the maximums of Equations (11) and (14) in the interval $0 \leq x \leq n$, we are interested in nonnegative integer solutions since the actual allocations must be nonnegative integers. In practice, these maximums are easy to find--either by using Newton's method and then rounding to integer answers (this must be done with care) or by performing a search over integer values of x between 0 and n .

Examples: (a) Constant Damage Rates. If we take $b(u) \equiv b$, $a(u) \equiv a$, and $c(u) \equiv c$ for $u \geq 0$, where $b > 0$, $a > 0$, and $c > 0$, then

$$D_B(x) = v_b(1 - e^{-bx}) ;$$

$$D_A(y) = v_a(1 - e^{-ay}) ; \text{ and}$$

$$N(w) = me^{-cw} .$$

In this case, the damage difference is given by

$$D(x, w) = v_b[1 - e^{-bx}] - v_a[1 - \exp(-aume^{-cw})] ;$$

and the damage ratio is given by

$$R(x, w) = v_b v_a^{-1} [1 - e^{-bx}] [1 - \exp(-aume^{-cw})]^{-1} .$$

(b) Variable Damage Rates. Suppose that $b(u) = bu^{\beta-1}$, where $b > 0$ and $\beta > 0$. Substituting this expression into Equation (3), we deduce that

$$D_B(x) = v_b [1 - \exp(-bx^\beta)] \quad (15)$$

for $x \geq 0$.

The case $0 < \beta < 1$ is very interesting; it corresponds to a decreasing damage rate, which is probably more realistic than a constant damage rate--since the first warheads targeted against B's economy are likely to be targeted against large industrial complexes that have high productive capability. After many warheads have been detonated against B's value targets, any additional warheads will attack either targets that have low values or targets that have already been destroyed. As a result, the fraction of B's remaining economy that is destroyed by these additional warheads will be lower than it was for the first few warheads that were detonated.

If $a(u) = au^{\alpha-1}$, where $a > 0$ and $\alpha > 0$, then substitution into Equation (5) yields

$$D_A(y) = v_a \left[1 - \exp(-ay^\alpha) \right]$$

for $y \geq 0$. Combining this result with Equation (15), we find that the damage difference in this case has the form

$$D(x,w) = v_b \left[1 - \exp(-bx^\beta) \right] - v_a \left[1 - \exp(-a[\mu N(w)]^\alpha) \right].$$

If $N(w) = me^{-cw}(c(u) \equiv c, c > 0)$, then this expression reduces to

$$D(x,w) = v_b \left[1 - \exp(-bx^\beta) \right] - v_a \left[1 - \exp(-a\mu^\alpha m^\alpha e^{-\alpha cw}) \right].$$

D. REMARKS

We conclude this section with two remarks concerning the appropriateness of the MOEs given by Equations (10) and (13). First, it is obvious from the latter that the ratio $v_b v_a^{-1}$ is just a multiplicative constant. Thus, the relative sizes of v_a and v_b do not affect the choice of the optimal allocation if the maximum damage ratio criterion of Equation (12) is used to determine A's allocation. If v_b is factored from the damage-difference expression given by Equation (10), we get

$$D(x,w) = v_b \left\{ \left[1 - \exp\left(-\int_0^x b(u)du\right) \right] - v_a v_b^{-1} \left[1 - \exp\left(-\int_0^{\mu N(w)} a(u)du\right) \right] \right\}.$$

From this expression, it is obvious that the ratio $v_a v_b^{-1}$ will affect the location of the optimum in Problem (11).

Second, the significance of the damage-difference criterion used in Expressions (10) and (11) needs some further interpretation. Let $\Delta_0 = v_a - v_b$ be the difference between the value

added of A and B before a nuclear war. Let

$$\Delta_1 = \Delta_1(x,w) = v_a A(\mu N(w)) - v_b B(x) \quad (16)$$

be the difference between the surviving value added of A and B after a nuclear war in which A chooses the allocation (x,w) and B retaliates with $\mu N(w)$ warheads. $A(y)$ is given by Equation (4) and $B(x)$ is given by Equation (2). If $\Delta_1 < \Delta_0$, then evidently A is the "loser"--since the difference in value added after the war has gone in the opposite direction from what A wanted. Comparing Equations (10) and (16), we deduce with the help of Equations (3) and (5) that

$$D(x,w) = \Delta_1(x,w) - \Delta_0 .$$

Thus, the difference in damage to B and maximum retaliatory damage to A equals the change in the difference in value added between A and B brought about by the nuclear war. With this interpretation in mind, it appears that the damage-difference criterion of Equation (10) is the better of the two measures given by Equations (10) and (13).

Chapter II

THE EFFECTS OF RANDOMNESS

A realistic treatment of a two-sided nuclear war must account for various random effects. The outcomes of many real-world activities are probabilistic rather than deterministic. A nuclear war is no exception. In fact, there are numerous sources of uncertainty for each side in a two-sided nuclear war. For discussion purposes, these sources of uncertainty can be divided into two types. The first type includes effects that are due to various physical characteristics of the weapon systems and targets. These random effects include the yield, reliability, and accuracy of the weapon systems; the hardness of targets; and the cumulative effects of these uncertainties on the level of damage to both weapon and value targets. These sources of randomness are amenable to mathematical treatment if certain reasonable assumptions are made; and we consider these random effects in some detail (below). Less amenable to mathematical analysis, the second type of uncertainty includes uncertain knowledge about the location of targets (in particular, movable weapon targets such as submarines and bombers) and the imperfect information that each side has about the other side's intentions and capabilities.

In this chapter, we consider the effects of the reliability of each side's warheads on the level of damage for both the first and second striker. We also analyze the effects of weapon yield, warhead accuracy, and weapon-target hardness on the number of B's weapon-launchers that survive A's first strike. Our objective is to develop correct expressions for the expected

first-strike damage to B and the expected maximum retaliatory damage to A. We shall also examine the variance of the damage difference and the effect of substituting expected values into the damage-difference function in place of the exact expected damage difference.

A. DERIVATION OF THE EXPECTED FIRST-STRIKE DAMAGE TO B

Suppose that A allocates x warheads against B's value system, where $0 \leq x \leq n$. Due to weapon system malfunctions only X_x warheads will actually detonate and damage B's economy. Obviously, we have $0 \leq X_x \leq x$. Assume that A's warheads have reliability r_a , where $0 < r_a < 1$. If we further assume that A's warheads function independently of one another once they have been allocated, then we get the following binomial distribution for X_x :

$$P(X_x = k) = \binom{x}{k} r_a^k (1-r_a)^{x-k} \quad k=0,1,2,\dots,x \quad (17)$$

The actual damage to B's economy is thus

$$D_B(X_x) = v_b [1 - B(X_x)] .$$

This quantity is a random variable, since X_x is a random variable. Using (17) we find that the expected damage to B's economy when A allocates x warheads, denoted by $E_B(x)$, is given by the formula

$$\begin{aligned} E_B(x) &= E[D_B(X_x)] = \sum_{k=0}^x D_B(k) P(X_x = k) \\ &= v_b \left[1 - \sum_{k=0}^x B(k) \binom{x}{k} r_a^k (1-r_a)^{x-k} \right] . \end{aligned} \quad (18)$$

For example, if $B(x) = e^{-bx}$ (corresponding to a constant damage rate $b(u) \equiv b, b > 0$), we find on substitution into (18) and

using the binomial theorem that

$$E_B(x) = v_b \left[1 - (r_a e^{-b} + 1 - r_a)^x \right] . \quad (19)$$

B. DERIVATION OF THE PROBABILITY OF A SINGLE A-WARHEAD KILLING A SINGLE B-WEAPON-LAUNCHER

Suppose that B's weapon-launchers have hardness H (measured in psi). In addition, assume that A's warheads have the same yield Y--measured in megatons--and the same CEP (circular error probable)--measured in nautical miles. Our objective is to derive an expression for the single-shot kill probability--denoted by p--for a single A-warhead killing a single B-weapon-launcher in terms of the parameters H, Y, and CEP. This derivation may be found in a recent article by K. Tsipis though there are mistakes in his derivation.¹ The derivation given here corrects these mistakes. According to Tsipis, the overpressure Δp (in psi) created at a distance r nautical miles from the point of detonation of a nuclear warhead with an explosive yield of Y megatons is given by the empirical formula

$$\Delta p = 14.7 \frac{Y}{r^3} + 12.8 \left(\frac{Y}{r^3} \right)^{\frac{1}{2}} . \quad (20)$$

If we let $\alpha = \left(\frac{Y}{r^3} \right)^{\frac{1}{2}}$ and divide by 12.8, then Equation (20) can be rewritten as a second-order polynomial in α :

$$\frac{\Delta p}{12.8} = 1.15\alpha^2 + \alpha .$$

Solving for α and retaining the positive root we get

$$\alpha = \frac{-1 + (1 + 0.36\Delta p)^{\frac{1}{2}}}{2.3} . \quad (21)$$

¹ "The Calculus of Nuclear Counterforce," *Technology Review* (Oct.-Nov. 1974), pp. 34-47. Equation (9a) on page 37 of Tsipis' article is incorrect. The correct equation is $P_s = \exp \left[- \left(\frac{r_s}{\text{CEP}} \right)^2 \ln 2 \right]$. Tsipis omits the factor $\ln 2$ here and in other equations in the article.

If Δp is large compared to 1 (say $\Delta p \geq 100$ psi), then we may neglect the 1 inside the radical in Equation (21) and get

$$\alpha = \left(\frac{Y}{r^3} \right)^{\frac{1}{2}} = -0.435 + 0.26(\Delta p)^{\frac{1}{2}} .$$

Squaring both sides of this equation and solving for r^3 , we obtain

$$\begin{aligned} r^3 &= \frac{Y}{\Delta p(0.19\Delta p^{-1} - 0.23\Delta p^{-\frac{1}{2}} + 0.068)} \\ &= \frac{Y}{\Delta p f(\Delta p)} , \end{aligned} \quad (22)$$

where $f(\Delta p) = 0.19\Delta p^{-1} - 0.23\Delta p^{-\frac{1}{2}} + 0.068$. Finally, taking the cube root of both sides of Equation (22), we get

$$r = \frac{Y^{\frac{1}{3}}}{(\Delta p)^{\frac{1}{3}} (f(\Delta p))^{\frac{1}{3}}} .$$

This equation gives the distance from the point of detonation at which the overpressure is Δp psi. Thus, if B's weapon-launchers have hardness H , then

$$r_s = \frac{Y^{\frac{1}{3}}}{H^{\frac{1}{3}} (f(H))^{\frac{1}{3}}} \quad (23)$$

is the distance (in nautical miles) from the point of detonation, at which $\Delta p = H$. Suppose now that a single A-warhead detonates at a distance r from a B-weapon-launcher. If $r < r_s$, then $\Delta p > H$; and the B-weapon-launcher will be destroyed. If $r \geq r_s$, then $\Delta p \leq H$, and the B-weapon-launcher will survive.

In order to compute the probability of an A-warhead killing a B-weapon-launcher, we need to know the probability that a single A-warhead detonates at a distance less than r_s from its aimpoint (the B-weapon-launcher in this case). In order to make this computation, we shall assume that the actual point of detonation (x,y) for A's warheads has a bivariate normal distribution centered about the aimpoint $(0,0)$. We assume further

that the x and y coordinates are independent. Under these assumptions, the distribution of the point of detonation has the density

$$f(x,y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2\sigma^2}(x^2+y^2)\right],$$

where σ is the standard deviation of the x and y coordinates.

Let p denote the probability that a single detonating A-warhead destroys a single B-weapon-launcher. Then

$$\begin{aligned} p = \text{Prob}(r \leq r_s) &= \iint_{\{u^2+v^2 \leq r_s^2\}} f(u,v) du dv \\ &= \int_0^{2\pi} \int_0^{r_s} \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right] r dr d\theta = 1 - \exp\left[-\frac{1}{2}\left(\frac{r_s}{\sigma}\right)^2\right]. \end{aligned} \quad (24)$$

By definition, the CEP of A's warheads is a distance (in nautical miles) such that

$$P(r \leq \text{CEP}) = \frac{1}{2}.$$

Substituting CEP in place of r_s in Equation (24), we see that

$$P(r \leq \text{CEP}) = 1 - \exp\left[-\frac{1}{2}\left(\frac{\text{CEP}}{\sigma}\right)^2\right] = \frac{1}{2}.$$

Solving for σ , we get

$$\sigma = \frac{\text{CEP}}{\sqrt{2\ln 2}}.$$

Substituting this expression into Equation (24) and rearranging, we get

$$p = 1 - \exp\left[-\ln 2 \left(\frac{r_s}{\text{CEP}}\right)^2\right]. \quad (25)$$

Finally, substituting Equation (23) into Equation (25) we obtain the kill probability in terms of Y , H , and CEP as follows

$$p = 1 - \exp\left[\frac{-Y^{\frac{2}{3}}\ln 2}{H^{\frac{2}{3}}(f(H))^{\frac{2}{3}}(CEP)^2}\right]. \quad (26)$$

Suppose now that A's warheads have reliability r_a . Then the probability that a single B-weapon-launcher survives an attack by a single A-warhead is $1-r_a + r_a(1-p)$, where p is given by Equation (26). Note that r_a is the probability that an A-warhead arrives in the vicinity of its target and *detonates*. Thus, with probability $1-r_a$, the warhead malfunctions and the target survives; with probability $r_a(1-p)$, the warhead detonates but the target survives. Finally, the probability that a single B-weapon-launcher is destroyed in an attack by ℓ A-warheads--denoted by p_ℓ --is given by

$$p_\ell = 1 - \left[1-r_a\right\{1 - \exp\left(\frac{-Y^{\frac{2}{3}}\ln 2}{H^{\frac{2}{3}}(f(H))^{\frac{2}{3}}(CEP)^2}\right)\right\}^\ell. \quad (27)$$

We note that this expression assumes that the ℓ warheads attacking a single B-weapon-launcher all act independently. Thus, Equation (27) neglects the interference effects caused by detonating more than one nuclear weapon near the target in rapid succession. Equation (27) allows us to calculate the probability that ℓ independent A-warheads will destroy a single B-weapon-launcher in terms of both the warhead parameters (r_a , Y , and CEP) and the target-hardness parameter (H).

C. DISTRIBUTION OF THE NUMBER OF B-WEAPON-LAUNCHERS SURVIVING AFTER A'S FIRST STRIKE

Let N_w denote the number of B-weapon-launchers that survive a first strike in which A allocates w warheads against B's weapon-launchers. Then $0 \leq N_w \leq m$, since B has m weapon-launchers. In order to derive a probability distribution for N_w , it is necessary to make some assumptions about the targeting

pattern that A uses in attacking B's weapon-launchers. Recall that we are assuming that the yield, reliability, and CEP are the same for all of A's warheads. Also, all of B's weapon-launchers are assumed to have the same hardness H , the same MIRV (Multiple Independently targeted Re-entry Vehicle) factor μ , and the same reliability r_b . In this situation, it is easy to show that A can maximize the number of B-weapon-launchers destroyed--and thereby minimize the maximum retaliatory damage that B can inflict--by distributing its w warheads uniformly over B's m weapon-launchers. Thus, the weapon-targeting scheme that we assume for A is as follows: Number B's weapon-launchers from 1 to m in any manner desired. For $w \geq 0$, let $\ell = \ell(w) = \left\lfloor \frac{w}{m} \right\rfloor$, where $[x]$ denotes the integer part of the nonnegative number x . Also, for $w \geq 0$, let

$$m_1 = m_1(w) = w - \ell m \quad \text{and} \quad m_2 = m_2(w) = m - m_1 = m(\ell+1) - w.$$

Then $m_1 \geq 0$, $m_2 \geq 0$, and $m_1 + m_2 = m$. We shall assume that A allocates $\ell+1$ warheads against B's first m_1 weapon-launchers and ℓ warheads against the remaining m_2 weapon-launchers.

Let M_1 denote the number of B's first $m_1 (=m_1(w))$ weapon-launchers that survive a first strike by $\ell+1$ A-warheads. Similarly, let M_2 denote the number of B's remaining $m_2 (=m_2(w))$ weapon-launchers that survive an attack by ℓ warheads. Then $N_w = M_1 + M_2$; and we have the following binomial distributions for M_1 and M_2 :

$$P(M_1=k) = \binom{m_1}{k} (p_{\ell+1})^{m_1-k} (1-p_{\ell+1})^k \quad k=0,1,2,\dots,m_1; \quad (28)$$

$$P(M_2=j) = \binom{m_2}{j} (p_{\ell})^{m_2-j} (1-p_{\ell})^j \quad j=0,1,2,\dots,m_2, \quad (29)$$

where p_{ℓ} is probability that a single B-weapon-launcher is destroyed in an attack by ℓ A-warheads and p_{ℓ} is given by Equation (27). Assuming that M_1 and M_2 are independent, we deduce that

$$P(N_w=i) = \sum_{k+j=i} P(M_1=k)P(M_2=j) \quad i=0,1,2,\dots,m. \quad (30)$$

This is the distribution that we sought for N_w .

D. COMPUTATION OF THE EXPECTED MAXIMUM RETALIATORY DAMAGE TO A

We assume now that B's surviving weapon-launchers have reliability r_b , where $0 < r_b < 1$. If a B-weapon-launcher functions correctly (with probability r_b), we assume that it delivers μ separate warheads to their respective targets and that each of these warheads detonate; if a surviving B-weapon-launcher malfunctions (with probability $1-r_b$), then we assume that none of its warheads are detonated.

Since $D_A(y)$ is increasing in y , it follows that B can maximize retaliatory damage to A by allocating all of its surviving weapons against A's value system. Let Y_w denote the number of B's remaining N_w weapon-launchers that function correctly in a retaliatory strike and, thus, successfully deliver μ warheads in retaliation against A's economy. Then $0 \leq Y_w \leq N_w$. Thus, the actual (maximum) retaliatory damage inflicted on A is $D_A(\mu Y_w)$, where $D_A(y)$ is given by Equation (5). Let $E_A(w)$ denote the expected maximum retaliatory damage to A's economy. Then

$$E_A(w) = E \left[D_A(\mu Y_w) \right].$$

Now let $G(N_w)$ denote the conditional expectation of $D_A(\mu Y_w)$, given N_w . Since the conditional distribution of Y_w , given N_w , is binomial with parameters N_w and r_b , we deduce that

$$G(N_w) = E \left[D_A(\mu Y_w) | N_w \right] = \sum_{k=0}^{N_w} D_A(\mu k) \binom{N_w}{k} r_b^k (1-r_b)^{N_w-k}. \quad (31)$$

Finally, taking the expectation of $G(N_w)$ with respect to the distribution given in Equation (30), we conclude that

$$\begin{aligned}
E_A(w) &= E[G(N_w)] = \sum_{i=0}^m G(i)P(N_w=i) \\
&= \sum_{i=0}^m \left\{ \sum_{k=0}^i D_A(\mu k) \binom{i}{k} r_b^k (1-r_b)^{i-k} \right\} \\
&\quad \times \left\{ \sum_{s+t=i} \left[\binom{m_1}{s} (p_{\ell+1})^{m_1-s} (1-p_{\ell+1})^s \binom{m_2}{t} (p_{\ell})^{m_2-t} (1-p_{\ell})^t \right] \right\}. \quad (32)
\end{aligned}$$

As an example of the use of Equation (32), suppose that $D_A(y) = v_a(1-e^{-ay})$. Substituting into Equation (32) and interchanging the order of summation, we find that

$$\begin{aligned}
E_A(w) &= \sum_{i=0}^m \left\{ \sum_{k=0}^i v_a(1-e^{-a\mu k}) \binom{i}{k} r_b^k (1-r_b)^{i-k} \right\} \\
&\quad \times \left\{ \sum_{s+t=i} \binom{m_1}{s} (1-p_{\ell+1})^s (p_{\ell+1})^{m_1-s} \binom{m_2}{t} (1-p_{\ell})^t (p_{\ell})^{m_2-t} \right\} \\
&= \sum_{i=0}^m v_a \left[1 - \left(r_b e^{-a\mu} + 1 - r_b \right)^i \right] \\
&\quad \times \left[\sum_{s+t=i} \binom{m_1}{s} (1-p_{\ell+1})^s (p_{\ell+1})^{m_1-s} \binom{m_2}{t} (1-p_{\ell})^t (p_{\ell})^{m_2-t} \right] \\
&= v_a - v_a \left\{ \left[\sum_{s=0}^{m_1} \left(r_b e^{-a\mu} + 1 - r_b \right)^s \binom{m_1}{s} (1-p_{\ell+1})^s (p_{\ell+1})^{m_1-s} \right] \right. \\
&\quad \times \left. \left[\sum_{t=0}^{m_2} \left(r_b e^{-a\mu} + 1 - r_b \right)^t \binom{m_2}{t} (1-p_{\ell})^t (p_{\ell})^{m_2-t} \right] \right\} \\
&= v_a - v_a \left\{ \left[p_{\ell+1} + (1-p_{\ell+1}) (r_b e^{-a\mu} + 1 - r_b) \right]^{m_1} \right. \\
&\quad \times \left. \left[p_{\ell} + (1-p_{\ell}) (r_b e^{-a\mu} + 1 - r_b) \right]^{m_2} \right\}. \quad (33)
\end{aligned}$$

The w dependence in this expression is rather complicated, since m_1 , m_2 and ℓ are functions of w .

E. RESTATEMENT OF THE MAX-MIN PROBLEM

Suppose now that A has n warheads and decides to launch a first strike against B. In order to evaluate the allocations chosen by A and B, we shall use the difference in expected damage as our measure of effectiveness. For this MOE, the model of the two-sided nuclear war can be stated as the following max-min problem:

$$\max_{\substack{x, w \geq 0 \\ x+w \leq n}} \min_{y \leq N_w} \left\{ E[D_B(X_x)] - E[D_A(\mu Y_y)] \right\}, \quad (34)$$

where Y_y denotes the actual number of B-weapon-launchers that function correctly when B retaliates with y weapon-launchers. Since $D_A(y)$ is increasing in y , it follows that $E[D_A(\mu Y_y)]$ is also increasing in y . Thus, the interior problem is solved by setting $y = N_w$. Since only $Y_w (= Y_{N_w})$ of B's weapon-launchers actually deliver their warheads, the maximum retaliatory damage is $D_A(\mu Y_w)$; and the expected maximum retaliatory damage is $E_A(w) = E[D_A(\mu Y_w)]$. Hence, the max-min problem in Expression (34) reduces to the following two-dimensional integer programming problem:

$$P_0: \begin{cases} \text{maximize } E(x, w) = E_B(x) - E_A(w) \\ \text{s.t. } x \geq 0, w \geq 0, x+w \leq n \quad \text{with } x \text{ and } w \text{ both integers,} \end{cases} \quad (35)$$

where $E_B(x)$ is given by Equation (18) and $E_A(w)$ is given by Equation (32). Since $D_B(x)$ is increasing in x , it follows that $E_B(x)$ is increasing in x . Also, it is possible to show that $E_A(w)$ is decreasing in w . Hence $E(x, w)$ is increasing in both x and w . We conclude from this fact that the solution to P_0 occurs at a point (x, w) such that $x + w = n$. Thus, P_0 reduces to the one-dimensional problem

$$P'_0: \begin{cases} \text{maximize } E(x, n-x) = E_B(x) - E_A(n-x) \\ x \in \{0, 1, \dots, n\} \end{cases} \quad (36)$$

By using Equations (18) and (32), this problem may be solved directly by searching over the set of integers from 0 to n .

F. APPROXIMATIONS

The use of Equations (18) and (32) for evaluating $E_B(x)$ and $E_A(w)$, respectively, can be time-consuming if x and m are large numbers, say $x \geq 1,000$ or $m \geq 1,000$. Therefore, it is worthwhile to inquire about approximations to $E_B(x)$ and $E_A(w)$.

Since X_x has the binomial distribution given by equation (17), we see that

$$E[X_x] = r_a x .$$

If x is large and r_a is not close to 0 or 1, then the distribution of X_x is almost normal with mean $r_a x$ and standard deviation $\sqrt{x r_a (1-r_a)}$. In this case, the standard deviation is small compared to the mean; and, using the normal approximation to the binomial distribution, we find that

$$P(|X_x - r_a x| \leq 3\sqrt{x r_a (1-r_a)}) \approx 0.98 .$$

If $D_B(x)$ is very flat in the neighborhood within three standard deviations on either side of the mean of X_x , then a useful approximation is to set

$$E_B(x) \approx D_B(E[X_x]) = D_B(r_a x) . \quad (37)$$

If $D_B(x)$ is concave for $x \geq 0$, we deduce with the help of Jensen's inequality that

$$E_B(x) = E[D_B(X_x)] \leq D_B(E[X_x]) = D_B(r_a x) .$$

Thus, when $D_B(x)$ is concave, it follows that the approximation $D_B(r_a x)$ overestimates the true expected damage to B.

Next, we seek an approximation to $E_A(w)$. Since the conditional distribution of Y_w , given N_w , is binomial with parameters N_w and r_b , we deduce that

$$E[Y_w | N_w] = r_b N_w .$$

Taking expectations on both sides of this equation, we find that

$$E[Y_w] = r_b E[N_w] .$$

However, $N_w = M_1 + M_2$; and the distributions of M_1 and M_2 are given by Equations (28) and (29), respectively. Hence,

$$E[N_w] = E[M_1] + E[M_2] = m_1(1-p_{\ell+1}) + m_2(1-p_{\ell}) .$$

Now let $g(w) = E[Y_w]$. Combining the above results, we see that

$$g(w) = r_b [m_1(1-p_{\ell+1}) + m_2(1-p_{\ell})] . \quad (38)$$

If $D_A(y)$ is concave for $y \geq 0$, then we deduce with the help of Jensen's inequality that

$$E_A(w) = E[D_A(\mu Y_w)] \leq D_A(\mu E[Y_w]) = D_A(\mu g(w)) . \quad (39)$$

Again, using the conditional distribution of Y_w , given N_w , we find that $\text{Var}(Y_w) = r_b^2 \text{Var}(N_w)$. Since M_1 and M_2 are assumed to be independent, we have

$$\begin{aligned} \text{Var}(N_w) &= \text{Var}(M_1) + \text{Var}(M_2) \\ &= m_1 p_{\ell+1} (1-p_{\ell+1}) + m_2 p_{\ell} (1-p_{\ell}) . \end{aligned}$$

Therefore, the standard deviation of Y_w is

$$r_b [m_1 p_{\ell+1} (1-p_{\ell+1}) + m_2 p_{\ell} (1-p_{\ell})]^{\frac{1}{2}} .$$

If m is large, this quantity is small compared to $g(w) (=E[Y_w])$.

Also, with probability close to 1, Y_w will be close to $g(w)$.

This is a consequence of the normal approximation to the binomial distribution. Thus, if $D_A(y)$ does not vary much in the neighborhood of $g(w)$, then we may also make the approximation

$$E_A(w) \simeq D_A(\mu g(w)) . \quad (40)$$

Combining the approximations made in Equations (37) and (40), we see that--when x and m are both large--we may approximate $E(x,w)$ by the function

$$E_1(x,w) = D_B(r_a x) - D_A(\mu g(w)) . \quad (41)$$

In this case, the allocation problem stated in Expression (36) may be replaced by the following problem:

$$P_1: \quad \underset{x \in \{0,1,2,\dots,n\}}{\text{maximize}} \quad E_1(x,n-x) . \quad (42)$$

When both sides have large numbers of weapons, the allocation determined by Expression (42) will be virtually the same as the allocation determined by Expression (36).

G. VARIANCE CALCULATIONS

It is interesting to consider the extent to which random variations can affect the actual damage difference. Assuming that B retaliates with all of its surviving weapon-launchers, then the actual damage difference is

$$D_B(X_x) - D_A(\mu Y_w) .$$

The expected value of this expression is the function $E(x,w)$ appearing in (35). In order to get some idea of the extent to which chance effects could alter the outcome of a two-sided nuclear war, we shall compute the variance of the damage difference for the special case in which both A and B have constant damage rate.

Suppose that $B(x) = e^{-bx}$ ($b > 0$) and $A(x) = e^{-ax}$ ($a > 0$). Then $D_B(x) = v_b(1-B(x)) = v_b(1-e^{-bx})$ and $D_A(y) = v_a(1-A(y)) = v_a(1-e^{-ay})$. Since an additive constant does not contribute to the variance of a random variable, we have

$$\text{Var}[D_B(X_x)] = v_b^2 \text{Var}[B(X_x)] = v_b^2 \text{Var}[\exp(-bX_x)] \quad (43)$$

and

$$\text{Var}[D_A(\mu Y_W)] = v_a^2 \text{Var}[A(\mu Y_W)] = v_a^2 \text{Var}[\exp(-a\mu Y_W)] . \quad (44)$$

From Equations (19) and (33) we have

$$E_B(x) = E[D_B(X_X)] = v_b [1 - (r_a e^{-b} + 1 - r_a)^x]$$

and

$$E_A(w) = E[D_A(\mu Y_W)] = v_a \left\{ 1 - [p_{\ell+1} + (1-p_{\ell+1})(r_b e^{-a\mu} + 1 - r_b)]^{m_1} \right. \\ \left. \times [p_{\ell} + (1-p_{\ell})(r_b e^{-a\mu} + 1 - r_b)]^{m_2} \right\} .$$

Thus, we see that

$$E[B(X_X)] = E[\exp(-bX_X)] = (r_a e^{-b} + 1 - r_a)^x \quad (45)$$

and

$$E[A(\mu Y_W)] = E[\exp(-a\mu Y_W)] = [p_{\ell+1} + (1-p_{\ell+1})(r_b e^{-a\mu} + 1 - r_b)]^{m_1} \\ \times [p_{\ell} + (1-p_{\ell})(r_b e^{-a\mu} + 1 - r_b)]^{m_2} . \quad (46)$$

By using Equation (17), it is easy to verify that

$$E[(B(X_X))^2] = (r_a e^{-2b} + 1 - r_a)^x .$$

Combining this result with Equation (45) and substituting into Equation (43), we deduce that

$$\text{Var}[D_B(X_X)] = v_b^2 [(r_a e^{-2b} + 1 - r_a)^x - (r_a e^{-b} + 1 - r_a)^{2x}] . \quad (47)$$

Next, by using the conditional distribution of Y_W , given N_W , we deduce that

$$\begin{aligned}
E\left[\left(A(\mu Y_w)\right)^2 | N_w\right] &= E\left[\exp(-2a\mu Y_w) | N_w\right] \\
&= \sum_{k=0}^{N_w} \exp(-2a\mu k) \binom{N_w}{k} r_b^k (1-r_b)^{N_w-k} \\
&= \left(r_b e^{-2a\mu + 1 - r_b}\right)^{N_w}.
\end{aligned}$$

Taking expectations with respect to the distribution of N_w , we find that

$$\begin{aligned}
E\left[\left(A(\mu Y_w)\right)^2\right] &= \left[p_{\ell+1} + (1-p_{\ell+1}) (r_b e^{-2a\mu + 1 - r_b})\right]^{m_1} \\
&\quad \times \left[p_{\ell} + (1-p_{\ell}) (r_b e^{-2a\mu + 1 - r_b})\right]^{m_2}.
\end{aligned}$$

Combining this result with Equation (46) and substituting into Equation (44), we conclude that

$$\begin{aligned}
\text{Var}\left[D_A(\mu Y_w)\right] &= v_a^2 \left\{ \left[p_{\ell+1} + (1-p_{\ell+1}) (r_b e^{-2a\mu + 1 - r_b})\right]^{m_1} \right. \\
&\quad \cdot \left[p_{\ell} + (1-p_{\ell}) (r_b e^{-2a\mu + 1 - r_b})\right]^{m_2} \\
&\quad - \left[p_{\ell+1} + (1-p_{\ell+1}) (r_b e^{-a\mu + 1 - r_b})\right]^{2m_1} \\
&\quad \cdot \left[p_{\ell} + (1-p_{\ell}) (r_b e^{-a\mu + 1 - r_b})\right]^{2m_2} \left. \right\}. \quad (48)
\end{aligned}$$

If A chooses the first strike allocation (x, w) and B retaliates with all of its surviving weapon-launchers, then the actual damage difference is

$$D_B(X_x) - D_A(\mu Y_w).$$

If we assume that X_x and Y_w are independent for each allocation (x, w) , then the variance of the damage difference is

$$\text{Var}\left[D_B(X_x) - D_A(\mu Y_w)\right] = \text{Var}\left[D_B(X_x)\right] + \text{Var}\left[D_A(\mu Y_w)\right]. \quad (49)$$

Thus, under the assumptions we have made, we may substitute Equations (47) and (48) into the right-hand side of Equation (49) to obtain the variance of the damage difference for this special case.

EXAMPLE:	Data for A	Data for B
	$a = 0.002$	$b = 0.003$
	$r_a = 0.8$	$r_b = 0.85$
	$p = 0.6$	$v_b = 10^{12}$ (dollars)
	$v_a = 10^{12}$ (dollars)	$m = 1,600$
	$x = 1,400$	$\mu = 3$
	$w = 1,600$	

In this example, we have $\lambda = \left[\frac{w}{m} \right] = \left[\frac{1,600}{1,600} \right] = 1$;
 $m_1 = w - \lambda m = 1,600 - 1,600 = 0$; and $m_2 = 1,600$. Thus,
 $p_1 = 1 - [1 - (0.8)(0.6)] = 0.48$. Using Equations (19) and (33),
 we find that the expected damage difference is

$$\begin{aligned}
 E(x, w) &= E_B(x) - E_A(w) \\
 &= 10^{12} \left\{ \left[1 - (0.8e^{-0.003+0.2})^{1,400} \right] \right. \\
 &\quad \left. - \left[1 - (0.48 + (0.52)(0.85e^{-0.006+0.15}))^{1,600} \right] \right\} \\
 &= -2.03 \times 10^{10} .
 \end{aligned}$$

Using Equations (47), (48), and (49), we find that the variance of the damage difference in this example is

$$\begin{aligned}
 &\text{Var}[D_B(X_x)] + \text{Var}[D_A(\mu Y_w)] \\
 &= 10^{24} \left\{ \left[(0.8e^{-0.006+0.2})^{1,400} - (0.8e^{-0.003+0.2})^{2,800} \right] \right. \\
 &\quad \left. + \left[0.48 + (0.52)(0.85e^{-0.012+0.15}) \right]^{1,600} \right\}
 \end{aligned}$$

$$= 10^{24} \times 5.435 \times 10^{-6} = 5.435 \times 10^{18} .$$

Taking the square root, we find that the standard deviation of the damage difference in this case is 2.331×10^9 dollars. Thus, in this example, the standard deviation of the damage difference is only 11 percent of the absolute value of the expected damage difference--which indicates that chance fluctuations of the damage difference about the expected damage difference are likely to be small in comparison to the expected damage difference.

Chapter III

DERIVATION OF A DAMAGE LAW FOR THE ECONOMY OF THE UNITED STATES

In this chapter, by fitting a curve to damage data generated in a previous IDA study, we derive a damage law for the economy of the United States. The data used here are from a second volume (unpubl.) of the June 1972 IDA Study S-394 (entitled *Methodologies for Evaluating Vulnerability of National Systems*, by James T. McGill et al.) and were supplied to the author by Dr. Leo Schmidt (of IDA). The data in Table 1 shows the U.S. value-added destroyed as a function of the number of one-megaton nuclear warheads detonated against the economy of the United States. For example, in an attack by 1,000 one-megaton weapons having a CEP of 0.5 nautical miles (reliability of 1.0 is assumed for the moment), it is estimated that the total value-added destroyed would be \$460.411 billion. The estimated total value added for the entire U.S. economy in 1975 is \$878.408 billion. Dividing the value-added destroyed by the total value-added of the economy, we obtain the fraction of value-added destroyed--denoted by $f(x)$. These data are shown in Table 1 as a function of the number of warheads (x) detonated in an attack on the U.S. economy. For example, for an attack with 1,000 one-megaton weapons, the fraction of value-added destroyed is $f(1,000) = 0.524$ (i.e., $\frac{460.411 \text{ billion}}{878.408 \text{ billion}} = 0.524$). Figure 1 shows an arithmetic scale graph of the fraction of value-added destroyed from Table 1 plotted versus the number of weapons used in an attack.

Table 1. U.S. VALUE-ADDED DESTROYED AS A FUNCTION OF ATTACK SIZE

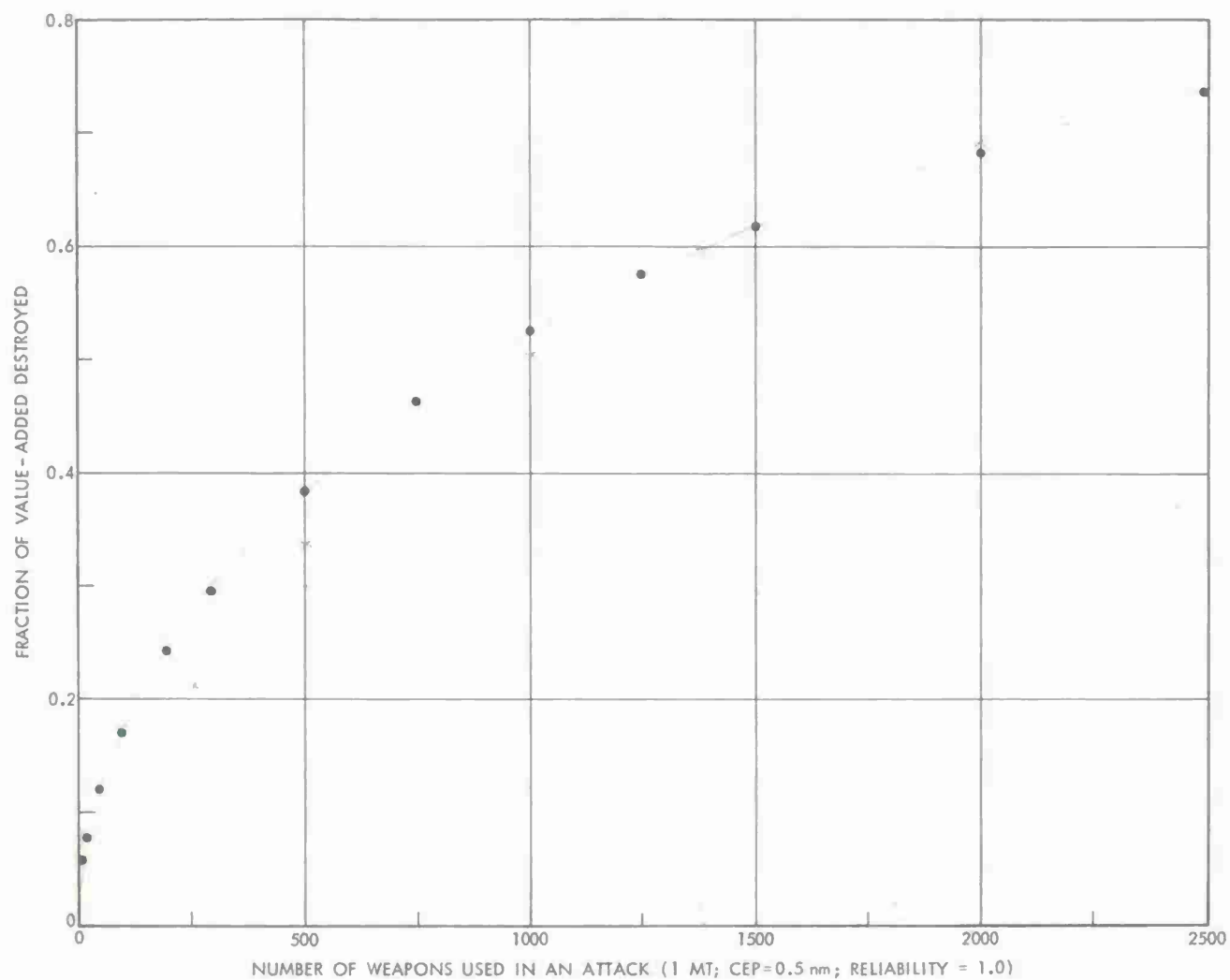
Number of Weapons ¹ x	Value-Added Destroyed (billions)	Fraction Destroyed ² f(x)	$-\ln(1-f(x))$	$v_b^{-1} D_B(x)$
10	49.946	0.057	0.05854	0.057
20	67.187	0.076	0.07957	0.076
50	105.019	0.120	0.12733	0.120
100	149.323	0.170	0.18632	0.171
200	212.594	0.242	0.27710	0.242
300	261.049	0.297	0.35266	0.297
500	335.652	0.382	0.48127	0.384
750	405.932	0.462	0.62012	0.464
1,000	460.411	0.524	0.74264	0.526
1,250	504.798	0.575	0.85490	0.576
1,500	541.787	0.617	0.95916	0.617
2,000	600.029	0.683	1.14913	0.683
2,500	644.511	0.734	1.32323	0.733
3,000	679.248	0.773	1.48400	0.772

¹Weapons are assumed to have 1-MT yield; CEP = 0.5 nm, and reliability = 1.0.
²For 1975, total value-added is taken to be \$878.408 billion.

Our objective in this section is to derive a damage law of the form

$$D_B(x) = v_b \left[1 - \exp\left(-\int_0^x b(u) du\right) \right] \quad (50)$$

for the United States, as a function of the number of one-megaton warheads (assumed to be allocated optimally) detonated in an attack on the U.S. economy. In Equation (50), v_b is the total value-added of the U.S. economy and $b(u)$ is the damage-rate function for the U.S. economy. Thus, the expression in brackets



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Figure 1. FRACTION OF U.S. VALUE-ADDED DESTROYED AS A FUNCTION OF ATTACK SIZE
[Estimates for 1975]

in Equation (50) is the fraction of value-added destroyed. Hence, we make the identification

$$f(x) = \left[1 - \exp\left(-\int_0^x b(u)du\right) \right]. \quad (51)$$

Taking logarithms in Equation (51), we see that

$$\int_0^x b(u)du = -\ln[1-f(x)] . \quad (52)$$

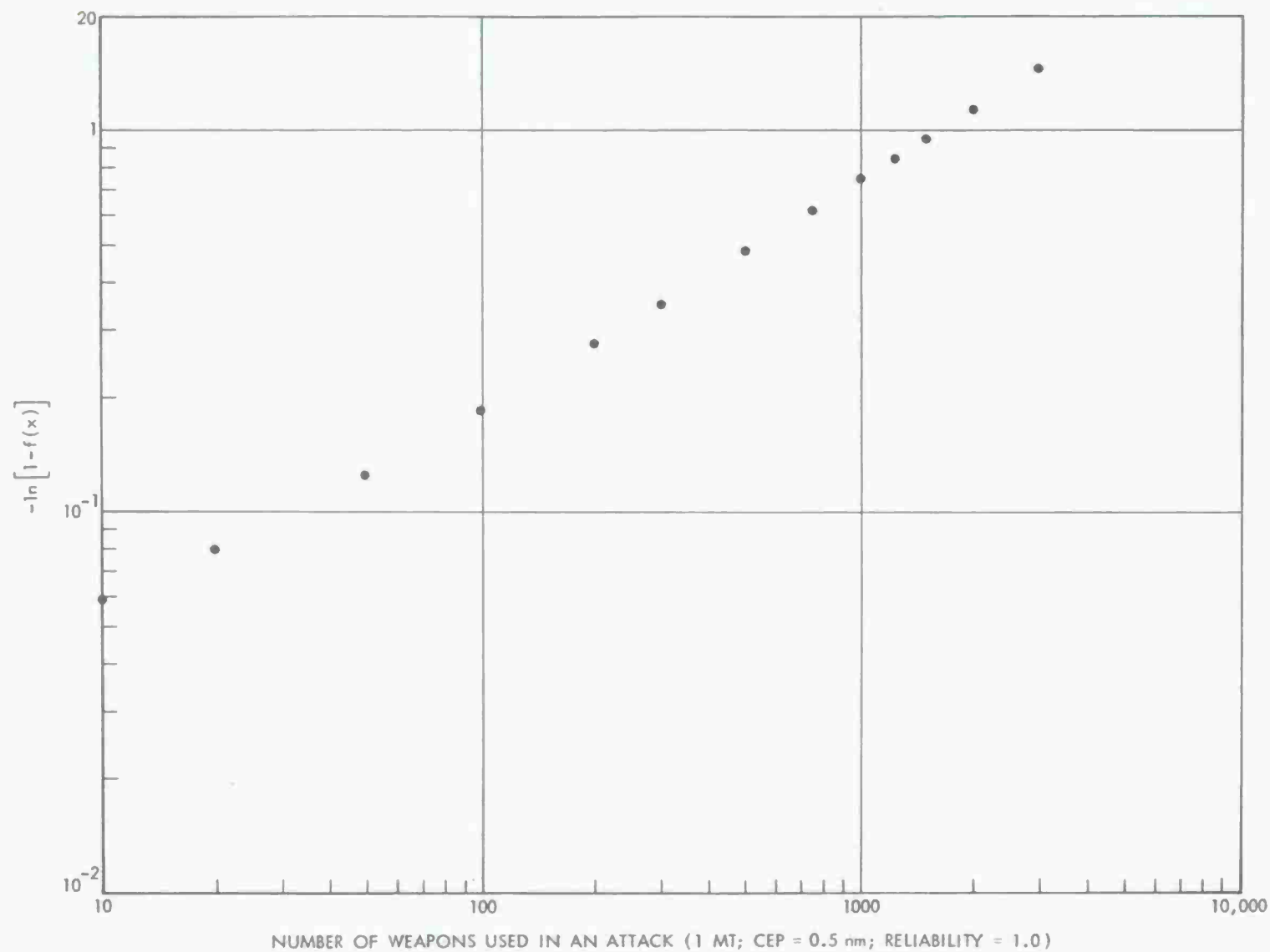
In Table 1 we have also computed values for the expression on the right-hand side of Equation (52). For $x = 1,000$, for example, we have

$$\int_0^{1,000} b(u)du = -\ln[1-f(1,000)] = -\ln[1-0.524] = 0.74264 .$$

Rather than attempting to estimate the damage-rate function $b(u)$, we shall instead estimate the integral

$$\int_0^x b(u)du ,$$

since this quantity is the argument of the exponential function in Equations (50) and (51). Figure 2 shows a full logarithmic plot of $-\ln[1-f(x)]$ from Table 1. We note that when the data are plotted on full logarithmic paper they fall approximately on a straight line. In fact, for $x \geq 300$ the data are almost precisely on a straight line. This fact immediately suggests that we estimate $\int_0^x b(u)du$ by using monomial functions of the form λx^α . After breaking the interval $[0,3000]$ into four sub-intervals and matching at the endpoints to obtain continuity, we obtained the following estimated form for the integral of the damage-rate function:



7-15-75-7

Figure 2. $-\ln[1-f(x)]$ FROM TABLE 1 PLOTTED VERSUS ATTACK SIZE

$$\int_0^x b(u)du = \begin{cases} (0.021179)x^{0.442802} & \text{for } 0 \leq x \leq 20, \\ (0.0171079)x^{0.513096} & \text{for } 20 \leq x \leq 50, \\ (0.0141889)x^{0.560917} & \text{for } 50 \leq x \leq 240.32, \\ (0.0101148)x^{0.622658} & \text{for } 240.32 \leq x. \end{cases} \quad (53)$$

The damage-rate function $b(u)$ can be found by differentiating the expressions on the right-hand side of Equation (53) with respect to x . Equation (53) may now be substituted into Equation (50) to obtain the damage function $D_B(x)$. The last column of Table 1 contains the result of substituting Equation (53) into Equation (50) and dividing by v_b . Comparing columns 3 and 5 of Table 1, we conclude that Equation (53) gives a very good fit to the hypothetical data on fraction of value-added destroyed.

The procedure we have used in deriving Equation (53) is completely empirical. We are given an estimate of the fraction of U.S. value-added destroyed for attacks of various sizes, and we want to derive a damage law for the U.S. economy as a function of the number of warheads detonated against the economy in some optimal manner. Equation (52) provides the appropriate link between the hypothetical data on fraction of value-added destroyed and the damage-rate function to be used in Equation (50). Since we have no data concerning the values of $b(x)$, it is better to estimate (i.e., fit a curve to) $\int_0^x b(u)du$ as a function of x than to try to estimate the damage-rate function $b(x)$. This procedure gives a better fit, since additional errors would be introduced by first estimating $b(x)$ and then integrating to obtain the argument of the exponential function in Equation (50). After all, it is the integral $\int_0^x b(u)du$ that is needed in Equation (50) as a function of x .

Data on fractional damage to the economy of the Soviet Union as a function of attack size are not available in unclassified documents. In order to proceed with the use of our model, we make a crude estimate for the integral of the damage rate for the Soviet economy--based upon the following heuristics: Since the manufacturing facilities of the Soviet Union are considered to be less concentrated than those of the United States, the damage rate for the economy of the Soviet Union should be smaller than the damage rate for the economy of the United States, at least for an attack involving hundreds of warheads. However, large Soviet cities tend to be more compact than large cities in the United States--for which reason a large city in the Soviet Union would be more heavily damaged in an attack by, say, a one-megaton warhead than would an American city (of corresponding size) attacked by the same one-megaton weapon. If large cities in the Soviet Union were targeted with higher priority than other industrial targets in the Soviet Union, the initial damage-rate for the Soviet economy would tend to be higher than that for the U.S. economy. Hence, we may estimate that (initially) the damage rate (not absolute damage) for the Soviet Union would be higher than that for the United States. At some point, however, the damage rate for the U.S. economy would exceed that for the Soviet Union.

For $x \geq 0$, let $a(x)$ denote the damage-rate function for the Soviet economy. We wish to choose an estimate of $\int_0^x a(u)du$ as a function of x . Based on the heuristic argument (above), the value of this integral should be greater than the integral in Equation (53) for small values of x . For large x , however, this integral should be less than the integral in Equation (53). In order to satisfy this requirement, we shall take $a(u) = 0.01x^{-\frac{1}{2}}$ for $x \geq 0$. Then

$$\int_0^x a(u)du = 0.02x^{\frac{1}{2}} . \quad (54)$$

Substituting (54) into Equation (5), we obtain the following expression for the hypothetical damage law for the Soviet Union:

$$D_A(x) = v_a [1 - \exp(-0.02\sqrt{x})] , \quad (55)$$

where v_a is the value (stated in value-added) of the Soviet economy. If this expression is evaluated (using $v_a = 1$) and compared with the values of $v_b^{-1}D_B(x)$ from Table 1, we see that

$$v_b^{-1}D_B(x) > 1 - \exp(-0.02x^{\frac{1}{2}})$$

for $x \geq 300$.

For the purpose of illustrating the use of the model, we use the expressions given by Equations (53) and (55). We also assume that the Soviet Union strikes first. We remark that if data of the type presented in Table 1 were available for the Soviet economy, then we could derive a damage law for the economy of the Soviet Union by using the same procedure that was used to derive Equation (53).

Chapter IV

NUMERICAL EXAMPLES

In this chapter, we examine several numerical examples of the aggregated two-sided nuclear-exchange model developed in the preceding sections. These examples include a hypothetical U.S.-Soviet nuclear war that uses the damage functions derived in Chapter 3 (above). These examples will also illustrate the sensitivity of the model to various physical parameters. In particular, we consider the special case in which B adopts a "launch on warning" policy.

The problem that we solve numerically in these examples is as follows:

$$P_2: \quad \underset{x \in \{0,1,2,\dots,n\}}{\text{maximize}} \quad v_b \left[1 - \exp \left(- \int_0^x b(u) du \right) \right] - v_a \left[1 - \left(\exp -0.02\sqrt{\mu g(n-x)} \right) \right]. \quad (56)$$

In this expression, $g(w)$ is the expected number of B-weapon-launchers retaliating against A after a first strike in which A attacks B's weapon-launchers with w warheads. $g(w)$ is given by the formula

$$g(w) = r_b \left[m_1(1-p_{\ell+1}) + m_2(1-p_\ell) \right],$$

where $\ell = \left\lceil \frac{w}{m} \right\rceil$; $m_1 = w - \ell m$; $m_2 = m - m_1$; and p_ℓ is given by Equation (27). The integral $\int_0^x b(u) du$ is evaluated using Equation (53).

The numerical procedure that we used to solve Problem P_2 is as follows: We evaluate the expression in Equation (56)

for $x = 1, 2, \dots, n$, and then find the value of x for which this expression is a maximum. This procedure works very well, even for large values of n --say, $n = 10,000$.

A. A HYPOTHETICAL NUCLEAR WAR BETWEEN THE UNITED STATES AND THE SOVIET UNION

We consider in this example a hypothetical nuclear war between the Soviet Union and the United States. We assume that the Soviet Union strikes first. The following values are assumed for the Soviet Union (A) and the United States (B):

<u>A</u>	<u>B</u>
$Y = 1.0$ MT	$H = 300$ psi
CEP = 0.5 nm	$r_b = 0.8$
$r_a = 0.7$	$\mu = 5$
$n = 3,000$	$m = 2,100$
$v_a = \$5 \times 10^{11}$ (value-added)	$v_b = \$8.78 \times 10^{11}$

With this choice of parameters, the probability of single-shot kill given by Equation (26) is $p = 0.347$. The optimum allocations in this example are $x = 3,000$; $w = 0$; and the maximum-damage difference is $\$1.879 \times 10^{11}$. The fraction of U.S. value-added destroyed is 0.69, and the fraction of Soviet value-added destroyed is 0.84. Since p is small in this case, it is not advantageous for the first striker to allocate any warheads against B's weapon-launchers.

B. THE EFFECTS OF A "LAUNCH ON WARNING" POLICY FOR B

From the previous example, we may infer that when the probability of A's warheads killing B's weapon-launchers is low, the effect on A's allocation is the same as if B had adopted a "launch on warning" policy. In other words, as long as the single-shot kill probability is small, the allocation is not sensitive to a "launch on warning" policy for the second striker.

However, when the probability of single-shot kill is close to 1, a possible strategy for modeling a "launch on warning" policy for B is to assume that the reliability of A's warheads that attack B's weapon-launchers is small--say less than 0.30. That is, we assume that A-warheads attacking B-weapon-launchers have reliability different from those attacking B's value targets. The effect of this assumption is that most of B's weapon-launchers survive and are able to retaliate. Let r_a^v denote the reliability of A's warheads attacking B's value targets and let r_a^w denote the reliability of A's warheads that attack B's weapon-launchers. Thus, we must substitute r_a^v in place of r_a in the integral in Equation (56); and we must substitute r_a^w in place of r_a in Equation (27). The following two cases illustrate this point.

Case 1.

<u>A</u>	<u>B</u>
Y = 1.0 MT	H = 300 psi
CEP = 0.1 nm	$r_b = 0.8$
$r_a^v = r_a^w = 0.8$	$\mu = 5$
n = 5,000	m = 2,100
$v_a = 1.0$	$v_b = 1.0$

For this case, the probability of single-shot kill determined by Equation (26) is $p = 0.999976$. The optimum allocation is $x = 2,900$; and $w = 2,100$. The fraction of A's economy destroyed is 0.56, and the fraction of B's economy destroyed is 0.72.

Case 2.

<u>A</u>	<u>B</u>
Y = 1.0 MT	H = 300 psi
CEP = 0.1 nm	$r_b = 0.8$
$r_a^v = 0.8$	$\mu = 5$

$$\begin{aligned}r_a^w &= 0.3 \\n &= 5,000 \\v_a &= 1.0\end{aligned}$$

$$\begin{aligned}m &= 2,100 \\v_b &= 1.0\end{aligned}$$

Since the parameters Y , CEP, and H are the same in this case as in Case 1, we again have $p = 0.999976$. The optimum allocation, however, is now $x = 5,000$; and $w = 0$. In this case, the fraction of A's economy destroyed is 0.84, and the fraction of B's economy destroyed is 0.83. Thus, even though A's warheads have high probability of destroying hard point targets (B's weapon targets), A chooses nevertheless to allocate all its warheads against B's economy. Also, in this case, the level of damage to both sides is considerably greater than in Case 1.

For Case 1 the maximum expected damage difference is 0.16 and for Case 2 it is -0.01. Thus, in Case 2 B improves his ability to retaliate by adopting a "launch on warning" policy. From these examples we conclude that when the CEP of A's warheads is small (say 0.1 nm or less), then B's deterrence is improved by adopting a "launch on warning" policy. However, such a policy results in a higher level of damage to each side.

C. THE EFFECT OF THE NUMBER OF WARHEADS ON A'S ALLOCATION

We now consider what effect the number of warheads at the disposal of the first striker has on his allocation. (In the remaining examples we take $r_a^v = r_a^w$ and denote the common value by r_a .) First, we consider the following two cases:

Case 3.

$$\begin{aligned}\underline{A} \\ Y &= 1.0 \text{ MT} \\ \text{CEP} &= 0.25 \text{ nm} \\ r_a &= 0.7\end{aligned}$$

$$\begin{aligned}\underline{B} \\ H &= 300 \text{ psi} \\ r_b &= 0.7 \\ \mu &= 5\end{aligned}$$

$$\begin{aligned} n &= 3,000 \\ v_a &= 1.0 \end{aligned}$$

$$\begin{aligned} m &= 2,100 \\ v_b &= 1.0 \end{aligned}$$

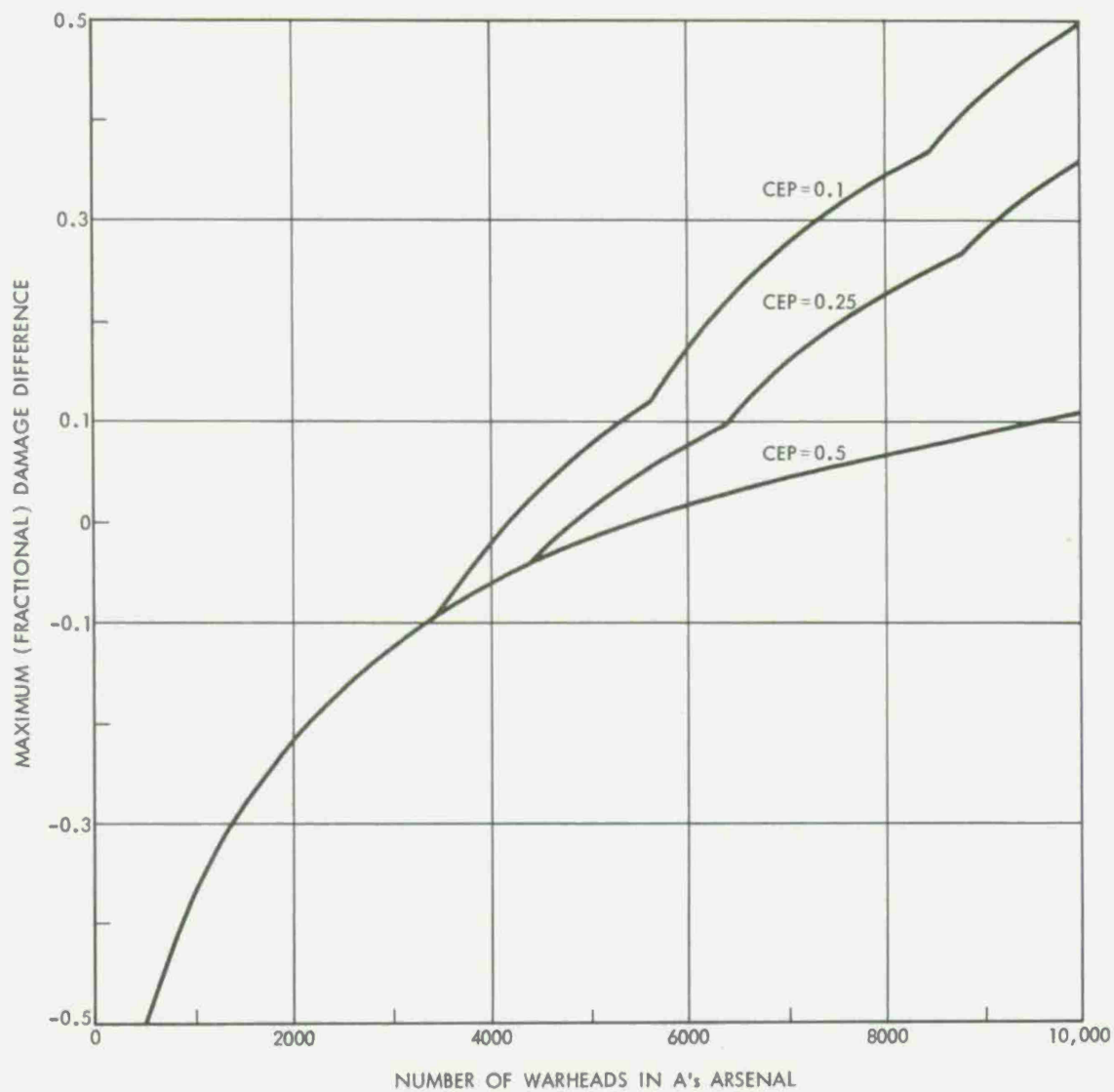
Case 4.

$$\begin{aligned} \underline{A} \\ Y &= 1.0 \text{ MT} \\ \text{CEP} &= 0.25 \text{ nm} \\ r_a &= 0.7 \\ n &= 5,000 \\ v_a &= 1.0 \end{aligned}$$

$$\begin{aligned} \underline{B} \\ H &= 300 \text{ psi} \\ r_b &= 0.7 \\ \mu &= 5 \\ m &= 2,100 \\ v_b &= 1.0 \end{aligned}$$

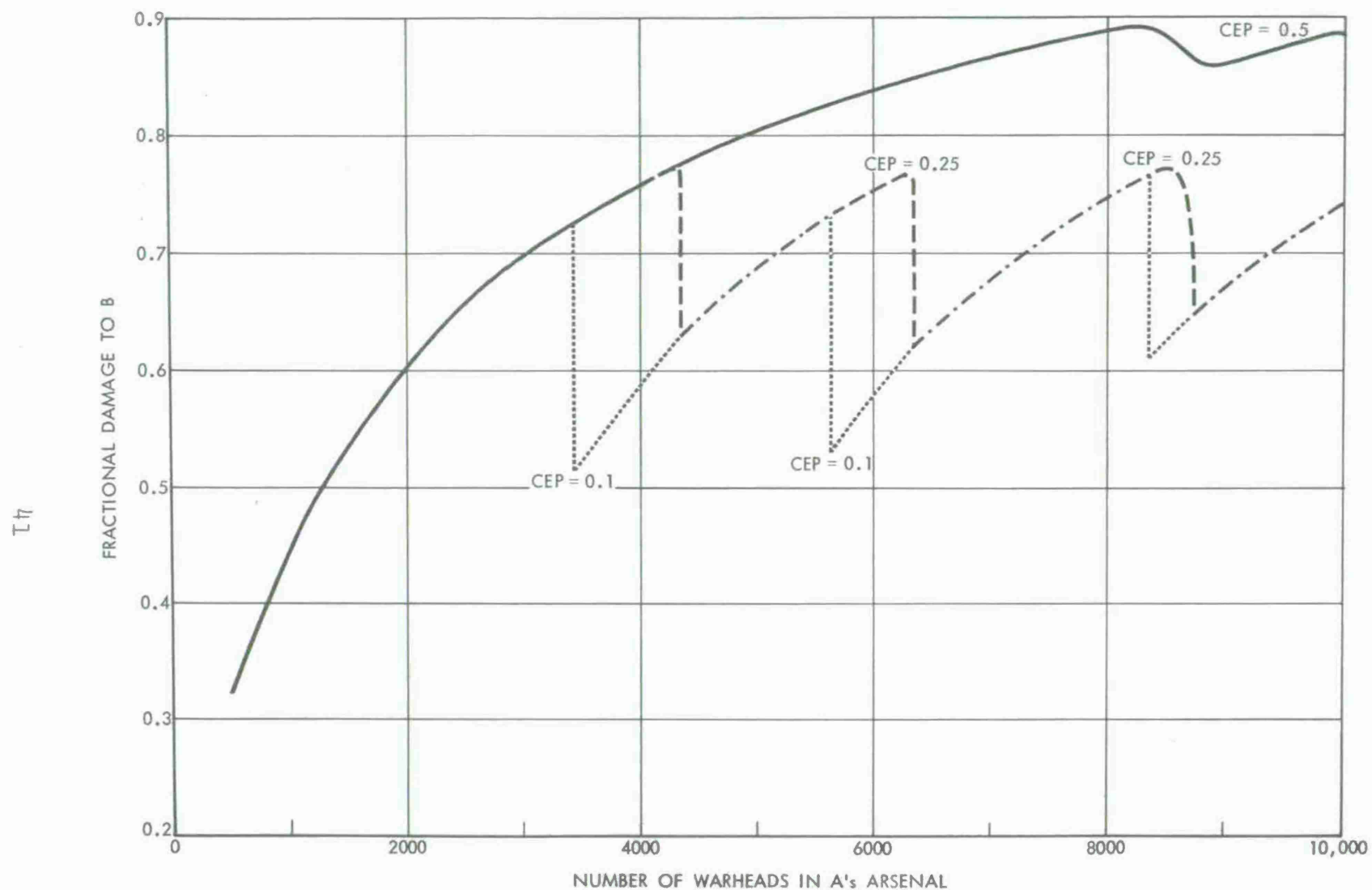
In each of these cases, the probability of single-shot kill is $p = 0.818$. For Case 3, the optimum allocation for A is $x = 3,000$; and $w = 0$. In this case, the fraction of A's economy destroyed is 0.82, and the fraction of B's economy destroyed is 0.69. For Case 4, the optimum allocation for A is $x = 2,900$; and $w = 2,100$. For Case 4, the fraction of A's economy destroyed is 0.67, and the fraction of B's economy destroyed is 0.69. It is interesting to note that the fractional damage to both sides in Case 4 is lower than in Case 3, even though the first striker has 5,000 warheads in Case 4 and only 3,000 warheads in Case 3.

Figures 3, 4, and 5 display graphically the results of a series of calculations in which we varied the number and CEP of A's warheads and held all the other variables constant. In each of these figures the number of warheads on A's side ranges from 500 to 10,000. For the CEP of A's warheads, we used the values 0.5, 0.25, and 0.1 nautical miles. The other variables are as follows:



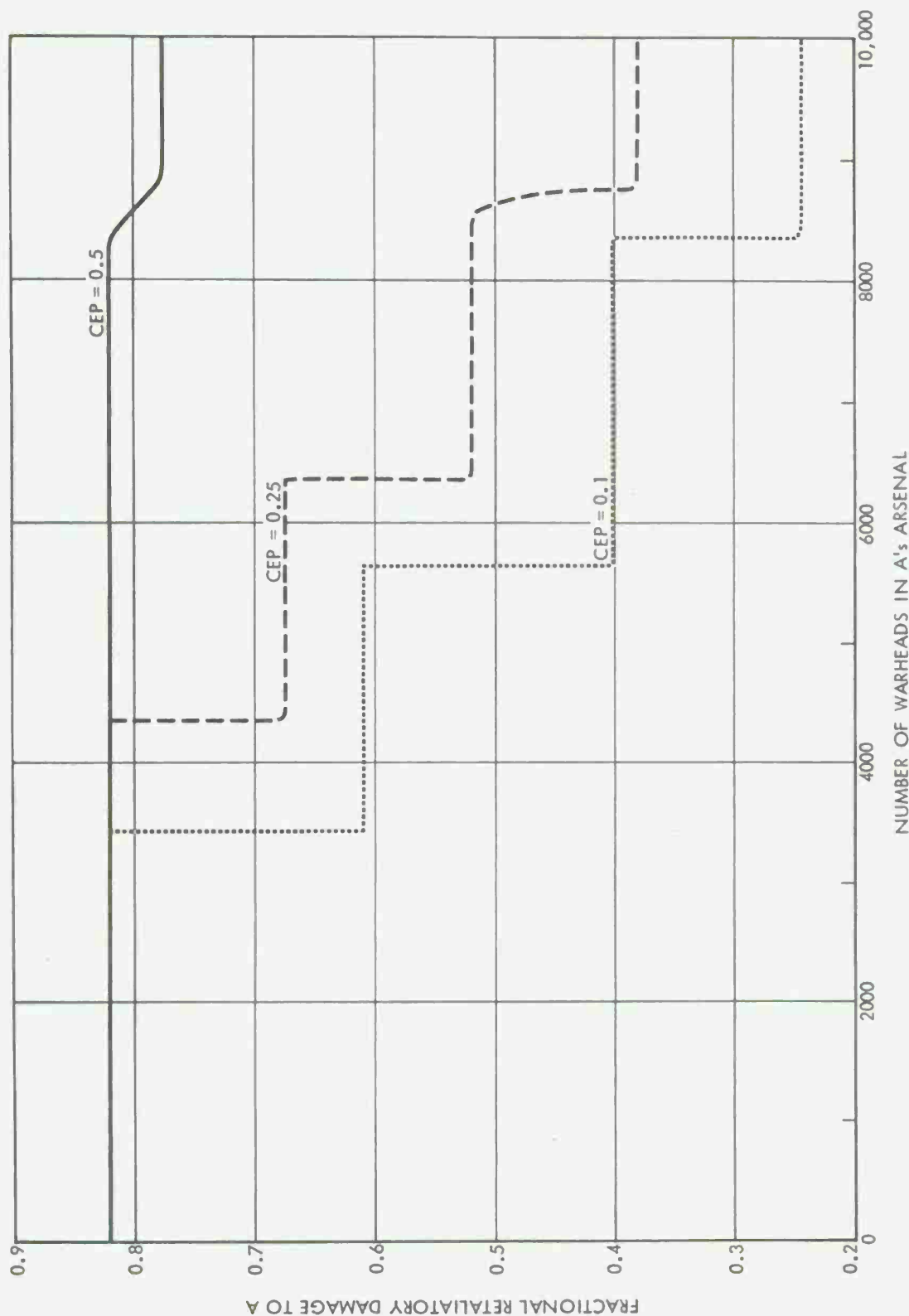
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Figure 3. MAXIMUM (FRACTIONAL) DAMAGE DIFFERENCE AS A FUNCTION OF THE SIZE OF A's NUCLEAR ARSENAL FOR THREE VALUES OF THE CEP. THE OTHER PARAMETERS ARE: $Y = 1.0$ MT; $r_a = 0.7$; $v_a = 1.0$; $H = 300$ psi; $r_b = 0.7$; $\mu = 5$; $m = 2,100$; AND $v_b = 1.0$.



7-15-75-9

Figure 4. FRACTION OF B'S ECONOMY DESTROYED AS A FUNCTION OF THE SIZE OF A'S FIRST STRIKE ATTACK FOR THREE VALUES OF THE CEP. THE OTHER PARAMETERS ARE: $Y = 1.0$ MT; $r_a = 0.7$; $v_a = 1.0$; $H = 300$ psi; $r_b = 0.7$; $\mu = 5$; $m = 2,100$; AND $v_b = 1.0$.



7-15-75-10

Figure 5. FRACTIONAL RETALIATORY DAMAGE TO A'S ECONOMY AS A FUNCTION OF THE SIZE OF A'S FIRST STRIKE ATTACK FOR THREE VALUES OF THE CEP. THE OTHER PARAMETERS ARE: $\gamma = 1.0$ MT; $r_a = 0.7$; $v_a = 1.0$; $H = 300$ psi; $r_b = 0.7$; $\mu = 5$; $m = 2,100$; AND $v_b = 1.0$.

$$\begin{aligned} & \underline{A} \\ Y &= 1.0 \text{ MT} \\ r_a &= 0.7 \\ v_a &= 1.0 \end{aligned}$$

$$\begin{aligned} & \underline{B} \\ H &= 300 \text{ psi} \\ r_b &= 0.7 \\ \mu &= 5 \\ m &= 2,100 \\ v_b &= 1.0 \end{aligned}$$

Figure 3 shows the maximum value of the (fractional) damage difference as a function of the size of A's first strike attack for the three different values of the CEP. These values were obtained by solving Problem P_2 of Equation (56). The three curves coincide for fewer than 3,400 warheads. The kinks in the curves for CEP = 0.25 and 0.1 occur at points where A's allocation against B's weapon-launchers increases by 2,100 warheads.

Figure 4 shows the fraction of B's economy destroyed as a function of the size of A's first strike attack for each of the three values of the CEP. Again, the three curves in Figure 4 coincide for fewer than 3,400 warheads; and the jumps in the curves occur at points where A's warhead allocation against B's weapon-launchers increases by 2,100 warheads.

Figure 5 shows the fraction of A's economy destroyed in a retaliatory strike by B as a function of the size of A's first strike attack for each of the three values of the CEP. We remark that the retaliatory damage to A decreases as the number and accuracy of A's warheads increases, since fewer B-weapon-launchers survive to retaliate.

Table 2 shows the allocations that yield the maximum damage difference in Figure 3. These allocations were found by solving Problem P_2 of Equation (56). For the case $N = 8,500$, we note that the number of warheads allocated against B's weapon-launchers is not an integral multiple of 2,100. We conjecture that this outcome is due to the fact that Problem P_2 is only an approximation to the exact maximum expected damage difference determined

Table 2. ALLOCATIONS YIELDING THE MAXIMUM DAMAGE DIFFERENCE
 FOR THE CASE $Y = 1.0$ MT; $r_a = 0.7$; $v_a = 1.0$;
 $H = 300$ psi; $r_b = 0.7$; $\mu = 5$; $m = 2,100$; $v_b = 1.0$

N	CEP = 0.5 nm		CEP = 0.25 nm		CEP = 0.1 nm	
	x	w	x	w	x	w
500	500	0	500	0	500	0
1,000	1,000	0	1,000	0	1,000	0
1,500	1,500	0	1,500	0	1,500	0
2,000	2,000	0	2,000	0	2,000	0
2,500	2,500	0	2,500	0	2,500	0
3,000	3,000	0	3,000	0	3,000	0
3,500	3,500	0	3,500	0	1,400	2,100
4,000	4,000	0	4,000	0	1,900	2,100
4,500	4,500	0	2,400	2,100	2,400	2,100
5,000	5,000	0	2,900	2,100	2,900	2,100
5,500	5,500	0	3,400	2,100	3,400	2,100
6,000	6,000	0	3,900	2,100	1,800	4,200
6,500	6,500	0	2,300	4,200	2,300	4,200
7,000	7,000	0	2,800	4,200	2,800	4,200
7,500	7,500	0	3,300	4,200	3,300	4,200
8,000	8,000	0	3,800	4,200	3,800	4,200
8,500	7,835	665	4,300	4,200	2,200	6,300
9,000	6,900	2,100	2,700	6,300	2,700	6,300
9,500	7,400	2,100	3,200	6,300	3,200	6,300
10,000	7,900	2,100	3,700	6,300	3,700	6,300

by Problem P_0 in Equation (35). We also conjecture that the optimum weapon allocation against B's weapon-launchers determined in Problem P_0 is an integral multiple of the number of B-weapon-launchers.

D. THE EFFECTS OF HARDNESS ON A'S ALLOCATION

We next consider what effect varying the hardness parameter H has on A's allocation and the maximum damage difference. Figure 6 shows the maximum value of the expected (fractional) damage difference for a series of calculations in which we varied the hardness of B's weapon-launchers and the number of A's warheads and held all the other variables constant. For the hardness of B's weapon-launchers we used the values 300, 600, and 1,000 psi. The number of warheads on A's side ranged from 500 to 10,000. The other variables were as follows:

<u>A</u>	<u>B</u>
$Y = 1.0$ MT	$r_b = 0.7$
CEP = 0.25 nm	$\mu = 5$
$r_a = 0.7$	$m = 2,100$
$v_a = 1.0$	$v_b = 1.0$

Table 3 shows the allocations that yield the maximum damage difference in Figure 3 for the three different values of H . These allocations were found by solving Problem P_2 of Equation (56).

When similar series of computations were performed using the values CEP = 0.1 nm and 0.5 nm, the maximum damage difference curves for the three different values of H virtually coincided. This indicates that when $CEP \leq 0.1$ nm or $CEP \geq 0.5$ nm, the effects of hardness on A's allocation and the maximum damage difference are negligible. This conclusion also seems intuitively reasonable. If $CEP \leq 0.1$ nm, then, with high probability, the attacking warhead detonates within a few hundred feet of its target. In this case, the target (a B-weapon-launcher) will almost certainly be destroyed no matter how hard it is. On the other hand, if $CEP \geq 0.5$ nm, then the weapon-launcher survives with high probability when $H = 300$ psi; and further hardening does not significantly enhance B's ability to retaliate.

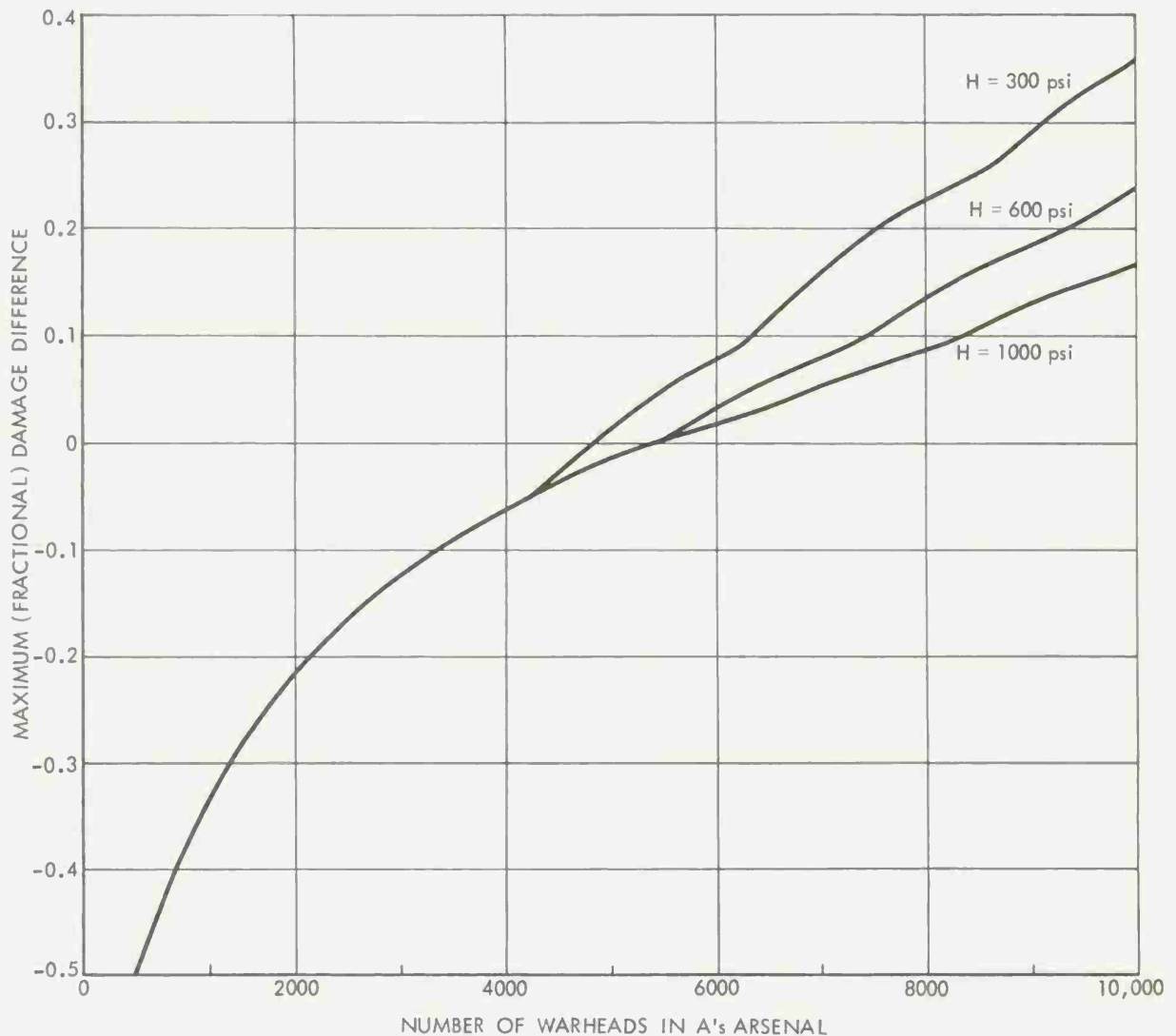
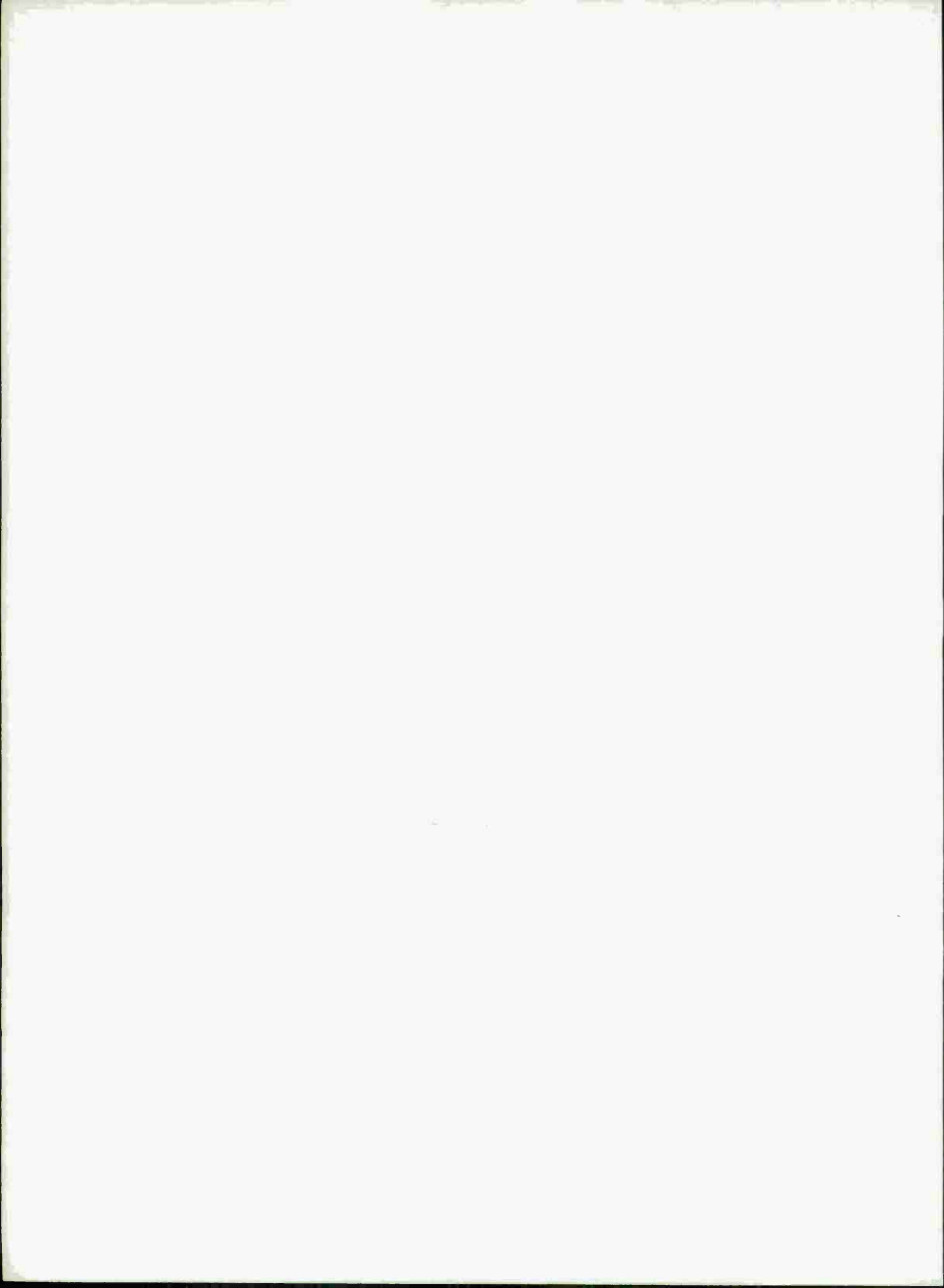


Figure 6. MAXIMUM (FRACTIONAL) DAMAGE DIFFERENCE AS A FUNCTION OF THE SIZE OF A'S NUCLEAR ARSENAL FOR THREE VALUES OF THE HARDNESS PARAMETER H . THE OTHER PARAMETERS ARE: $Y = 1.0$ MT; $CEP = 0.25$ nm; $r_a = 0.7$; $v_a = 1.0$; $r_b = 0.7$; $\mu = 5$; $m = 2,100$; and $v_b = 1.0$.

Table 3. ALLOCATIONS YIELDING THE MAXIMUM DAMAGE DIFFERENCE
FOR THE CASE $Y = 1.0$ MT; $CEP = 0.25$ nm; $r_a = 0.7$;
 $v_a = 1.0$; $r_b = 0.7$; $\mu = 5$; $m = 2,100$; and $v_b = 1.0$

N	H = 300 psi		H = 600 psi		H = 1,000 psi	
	x	w	x	w	x	w
500	500	0	500	0	500	0
1,000	1,000	0	1,000	0	1,000	0
1,500	1,500	0	1,500	0	1,500	0
2,000	2,000	0	2,000	0	2,000	0
2,500	2,500	0	2,500	0	2,500	0
3,000	3,000	0	3,000	0	3,000	0
3,500	3,500	0	3,500	0	3,500	0
4,000	4,000	0	4,000	0	4,000	0
4,500	2,400	2,100	4,500	0	4,500	0
5,000	2,900	2,100	5,000	0	5,000	0
5,500	3,400	2,100	3,400	2,100	5,500	0
6,000	3,900	2,100	3,900	2,100	6,000	0
6,500	2,300	4,200	4,400	2,100	6,500	0
7,000	2,800	4,300	4,900	2,100	4,900	2,100
7,500	3,300	4,200	3,300	4,200	5,400	2,100
8,000	3,800	4,200	3,800	4,200	5,900	2,100
8,500	4,300	4,200	4,300	4,200	4,413	4,087
9,000	2,700	6,300	4,800	4,200	4,800	4,200
9,500	3,200	6,300	4,032	5,468	5,300	4,200
10,000	3,700	6,300	3,700	6,300	5,800	4,200



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